Inequality 11

19 November 2023 16:04

Cauchy-Schwarz Inequality :-

For real numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ we get,

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Equality holds iff $\exists$ some $\lambda \in \mathbb{R}$ such that $x_i = \lambda y_i$ $\forall\, i \in \{1, 2, \dots, n\}$

Proof :-

maximum value

WLOG we can assume here $z_1 < z_2 < z_3 \quad < z_n$

$$2 = \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} + \frac{\sum_{i=1}^{n} y_i^2}{\sum_{i=1}^{n} y_i^2} = \frac{\sum_{i=1}^{2n} z_i^2}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

$$\left.\begin{aligned} z_i^2 &= \left(\sum y_i^2\right) x_i^2 \\ z_{n+i}^2 &= \left(\sum x_i^2\right) y_i^2 \end{aligned}\right\} i = 1, \dots, n$$

$$\geq \frac{\left(z_1 z_{n+1} + z_2 z_{n+2} + \dots + z_n z_{2n}\right) + \left(z_{n+1} z_1 + z_{n+2} z_2 + \dots + z_{2n} z_n\right)}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

$\hookrightarrow$ Rearrangement Inequality

$$= 2 \frac{\sum \left(z_i z_{n+i}\right)}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

$$= 2 \frac{\sum\left(\left(\sqrt{\sum y_i^2}\, x_i\right)\left(\sqrt{\sum x_i^2}\, y_i\right)\right)}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

$$= 2 \frac{\left(\sqrt{\sum y_i^2}\right)\left(\sqrt{\sum x_i^2}\right) \sum x_i y_i}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

$$= 2 \frac{\sum x_i y_i}{\sqrt{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}}$$

[illegible] $\frac{\sum x_i y_i}{\dots}$ $\Rightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$

$$\Rightarrow \quad 2 \geq 2 \frac{\sum x_i y_i}{\sqrt{(\sum x_i^2)(\sum y_i^2)}} \quad \Rightarrow \quad \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Q) Let a, b, c be positive numbers with $a+b+c=1$, prove that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$

Ans :- $\left(\frac{1-a}{a}\right) = \frac{b+c}{a} \qquad \frac{1-b}{b} = \frac{a+c}{b} \qquad \frac{1-c}{c} = \frac{a+b}{c}$

$$\Rightarrow \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right)\left(\frac{a+b}{c}\right)$$

<u>Nesbitt's Inequality</u> :-

For $a, b, c \in \mathbb{R}^+$ we have,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

<u>Proof</u> :- WLOG, $a \leq b \leq c$

$(b+c)-(a+c) = b-a \geq 0$

$(a+c)-(a+b) = c-b \geq 0$

$$\Rightarrow a+b \leq a+c \leq c+b$$

$$\Rightarrow \frac{1}{c+b} \leq \frac{1}{a+c} \leq \frac{1}{a+b}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \quad \text{— (1)}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b} \quad \text{— (2)}$$

(1)+(2)

$$\Rightarrow 2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{c+a}{c+a} + \frac{b+a}{a+b} = 3$$

① + ②

$$\Rightarrow 2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq \frac{b+c}{b+c}+\frac{c+a}{c+a}+\frac{b+a}{a+b} = 3$$

$$\Rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

Q> Let $a, b, c \in \mathbb{R}^+$ with $abc = 1$ Prove that,

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Ans:— WLOG, $a \leq b \leq c$ $\qquad abc = 1 \qquad x = \frac{1}{a} \quad y = \frac{1}{b} \quad z = \frac{1}{c}$

$z \leq y \leq x \qquad xyz = 1$

$x, y, z \in \mathbb{R}^+$

$$\frac{1}{a^3(b+c)} = \frac{x^3}{\frac{1}{y}+\frac{1}{z}} = \frac{x^3yz}{y+z} = \frac{x^2(xyz)}{y+z} = \frac{x^2}{y+z} \quad (xyz=1)$$

$$\frac{1}{b^3(c+a)} = \frac{y^2}{z+x} \qquad \frac{1}{c^3(a+b)} = \frac{z^2}{x+y}$$

$$LHS = \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}$$

$z \leq y \leq x$

$\Rightarrow y+z \leq z+x \leq x+y$

$\Rightarrow \frac{1}{x+y} \leq \frac{1}{z+x} \leq \frac{1}{y+z}$

$\Rightarrow \frac{z}{x+y} \leq \frac{y}{z+x} \leq \frac{x}{y+z}$

$$LHS \geq \frac{zy}{x+y}+\frac{xy}{z+x}+\frac{xz}{y+z} \quad —①$$

$$LHS \geq \frac{xz}{x+y}+\frac{yz}{z+x}+\frac{xy}{y+z} \quad —②$$

① + ②

$$\Rightarrow 2(LHS) \geq \frac{z(x+y)}{x+y}+\frac{y(x+z)}{x+z}+\frac{x(z+y)}{y+z} = z+x+y \geq 3\sqrt[3]{xyz} = 3$$

AM-GM

$\rightarrow$ AM-GM

$\Rightarrow$ LHS $\geqslant \frac{3}{2}$

Q) Let $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ such that $a_i \in [0, i]$ $\forall i \in \{1, 2, \dots, n\}$

Prove that,

$$2^n a_1 (a_1+a_2)(a_1+a_2+a_3) \cdots (a_1+a_2+\cdots+a_n) \geq (n+1) a_1^2 a_2^2 \cdots a_n^2$$

$$a_1 \in [0,1]$$
$$a_2 \in [0,2]$$
$$a_3 \in [0,3]$$
$$\vdots$$

Ans:- $a_1 + a_2 + \cdots + a_k = a_1 + \frac{a_2}{2} + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_3}{3} + \frac{a_3}{3} + \cdots + \underbrace{\frac{a_k}{k} + \cdots + \frac{a_k}{k}}_{k \text{ times}}$

$$\geqslant \frac{k(k+1)}{2} \sqrt[\frac{k(k+1)}{2}]{a_1 \left(\frac{a_2}{2}\right)^2 \left(\frac{a_3}{3}\right)^3 \cdots \left(\frac{a_k}{k}\right)^k}$$

$$\prod_{k=1}^{n} (a_1 + \cdots + a_k) \geqslant \prod_{k=1}^{n} \left(\frac{k(k+1)}{2} \sqrt[\frac{k(k+1)}{2}]{a_1 \left(\frac{a_2}{2}\right)^2 \left(\frac{a_3}{3}\right)^3 \cdots \left(\frac{a_k}{k}\right)^k} \right)$$

$$= \left(\prod_{k=1}^{n} \frac{1}{2}\right)\left(\prod_{k=1}^{n} k\right)\left(\prod_{k=1}^{n} (k+1)\right)\left(\prod_{k=1}^{n} \left(\frac{a_k}{k}\right)^{f(k)}\right)$$

$$f(k) = k\left(\frac{2}{k(k+1)} + \frac{2}{(k+1)(k+2)} + \cdots + \frac{2}{n(n+1)}\right)$$

$$= 2k\left(\frac{1}{k} - \frac{1}{n+1}\right) = 2\left(1 - \frac{k}{n+1}\right) \leq 2$$

$$\prod_{k=1}^{n} (a_1 + \cdots + a_k) \geqslant \left(\frac{1}{2}\right)^n n!\,(n+1)! \prod_{k=1}^{n} \left(\left(\frac{a_k}{k}\right)^2\right)$$

$$\prod_{k=1}^{n} \left(\frac{1}{k^2}\right) = \prod_{k=1}^{n} \left(\frac{1}{k}\right) \prod_{k=1}^{n} \left(\frac{1}{k}\right)$$

$$= \frac{1}{1 \times 2 \times \cdots n} \times \frac{1}{1 \times 2 \times 3 \cdots n}$$

$$= \frac{1}{n!} \times \frac{1}{n!}$$

$$\prod_{k=1} (a_1 + \cdots + a_k) \geq \left(\frac{\ }{2}\right) \cdots \quad 1\times 2\times \cdots n \quad 1\times 2\times 3\cdots n = \frac{1}{n!} \times \frac{1}{n!}$$

$$\Rightarrow \prod_{k=1}^{n} (a_1 + \cdots + a_k) \geq \frac{\cancel{n!}\,(n+1)\cancel{!}}{2^n\,\cancel{n!}\,\cancel{n!}}\, a_1^2 a_2^2 \cdots a_n^2 = \frac{n+1}{2^n}\, a_1^2 a_2^2 \cdots a_n^2$$

$$\Rightarrow 2^n (a_1)(a_1 + a_2) \cdots (a_1 + \cdots + a_n) \geq (n+1)\, a_1^2 a_2^2 \cdots a_n^2$$

Tchebyshev's Inequality: $a_i, b_i \in \mathbb{R}$

$a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ then,

$$\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right)$$

Equality holds iff $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Proof: $S = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ is max sum in rearrangement.

$$S \geq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$S \geq a_1 b_2 + a_2 b_3 + \cdots + a_n b_1$$

$$S \geq a_1 b_3 + a_2 b_4 + \cdots + a_n b_2$$

$$\vdots$$

$$S \geq a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1}$$

$$nS \geq a_1 (b_1 + \cdots + b_n) + a_2 (b_1 + b_2 + \cdots + b_n) + \cdots + a_n (b_1 + b_2 + \cdots + b_n)$$

$$\Rightarrow \frac{S}{n} \geq \frac{(a_1 + a_2 + \cdots + a_n)}{n} \frac{(b_1 + b_2 + \cdots + b_n)}{n}$$

Home Work :-

Q) $a, b, c \in \mathbb{R}^+$, then prove that :-

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Q) $x, y, z \geq 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$